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# Path Integral Approach to the Propagator for a Charged Particle in a Time-Dependent Electromagnetic Field[1]

Jing-Bo XU, Yu-Jun ZHAO and Xiao-Chun GAO
Department of Physics, Zhejiang University, Hangzhou 310027, China
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**Abstract**
*A path integral method is used to study the time evolution of a three-dimensional time-dependent system. The propagator for a charged particle in a time-dependent magnetic field and quadrupolar electric potential is obtained.*

The study of the time evolution of the time-dependent systems has long been of interest. The systems with the time-dependent Hamiltonians were investigated through various methods.[1–8] The recent discovery of quantum Hall effect and high-$T_c$ superconductivity has stimulated the interest of studying quantum mechanics of nonrelativistic particle moving in magnetic fields.[9–10] Using the $\zeta$-function method, Farina and Gamboa[10] obtained the propagator for a harmonically bound charged particle in a constant magnetic field. More recently, Gheorghe and Vedel[11] studied the time evolution of an ion in a trap with a constant magnetic field and a quadrupolar electric potential. In Refs [7] and [8], a path integral method was applied to the study of a one-dimensional system. In this paper, the method is generalized and used to study the time evolution of a three-dimensional time-dependent system. We find the exact propagator for a charged particle in a time-dependent magnetic field and quadrupolar electric potential. As the magnetic field becomes time-independent, the result obtained in the present paper is in agreement with that of Refs [10] and [11].

The Hamiltonian for a charged particle in a time-dependent magnetic field $\vec{B}(t) = B(t)\hat{e}_3$ ($\hat{e}_3$ is unit vector for 3-axis) and a quadrupolar electric potential $\phi(\vec{x},t) = A(t)(x_1^2 + x_2^2 - 2x_3^2)$ is

$$H(t) = \frac{[\vec{p} - q\vec{B}(t) \times \vec{x}]^2}{(2m)} + q\phi(\vec{x},t)\,, \tag{1}$$

where $\vec{x} = \sum_{j=1}^{3} x_j \hat{e}_j$ is the coordinate and $\vec{p} = \sum_{j=1}^{3} p_j \hat{e}_j$ is the momentum of a particle with charge $q$ and mass $m$. As the magnetic field becomes time-independent, the Hamiltonian (1) reduces to the Hamiltonian in Ref. [11]. The Hamiltonian (1) can be rewritten as

$$H(t) = H_1(t) + H_2(t)\,, \tag{2}$$

where

$$H_1(t) = \frac{p_3^2}{2m} + \frac{ma(t)x_3^2}{2}\,, \qquad a(t) = \frac{-4qA(t)}{m}\,, \tag{3}$$

$$H_2(t) = \frac{p_1^2 + p_2^2}{2m} + \frac{mb(t)(x_1^2 + x_2^2)}{2} + \frac{\omega(t)(x_1 p_2 - x_2 p_1)}{2}\,,$$
$$b(t) = \frac{-4qA(t)}{m} - \frac{qB(t)}{4m}\,, \qquad \omega(t) = \frac{-qB(t)}{m}\,, \tag{4}$$

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In the canonical formulation of the path integrals, the propagator of the system can be expressed as

$$K(\vec{x}_f,t_f;\vec{x}_i,t_i)=K_1(x_{3f},t_f;x_{3i},t_i)K_2(x_{2f},x_{1f},t_f;x_{2i},x_{1i},t_i)$$
$$=\int\prod_{j=1}^{3}\mathcal{D}x_j\mathcal{D}p_j\exp\Big\{\frac{i}{\hbar}\int_{t_i}^{t_f}\Big[\sum_{j=1}^{3}p_j\dot{x}_j-H_1(t)-H_2(t)\Big]dt\Big\}, \tag{5}$$

where $\dot{x}_j=dx_j/dt\,(j=1,2,3)$. Using the method in Refs [7] and [8], it is easy to obtain the propagator corresponding to the Hamiltonian $H_1(t)$

$$K_1(x_{3f},t_f;x_{3i},t_i)=\Big(\frac{m}{2i\hbar\pi\tau_1v_iv_f}\Big)^{1/2}\exp\Big[\frac{im}{2\hbar}\Big(\frac{\dot{v}_fx_{3f}^2}{v_f}-\frac{\dot{v}_ix_{3i}^2}{v_i}\Big)\Big]$$
$$\times\exp\Big[\frac{im}{2\hbar\tau_1}\Big(\frac{x_{3f}}{v_f}-\frac{x_{3i}}{v_i}\Big)^2\Big], \tag{6}$$

where

$$v_i=v(t_i),\qquad v_f=v(t_f),\qquad \tau_1=\int_{t_i}^{t_f}v^{-2}(t)dt$$

and $v(t)$ is the solution of the following equation

$$\ddot{v}(t)+a(t)v(t)=0. \tag{7}$$

We then try to find a transformation which can transform the system with Hamiltonian $H_2(t)$ into a two-dimensional free particle. Let us consider the following transformation which is the generalization of the canonical transformation in Refs [7] and [8]

$$(x_1,x_2,p_1,p_2,t)\longrightarrow(Q_1,Q_2,P_1,P_2,\tau),$$
$$x_1=u(t)[Q_1\cos\alpha(t)-Q_2\sin\alpha(t)],\quad x_2=u(t)[Q_1\sin\alpha(t)+Q_2\cos\alpha(t)],$$
$$p_1=\frac{[P_1\cos\alpha(t)-P_2\sin\alpha(t)]+m\dot{u}(t)x_1}{u(t)}, \tag{8}$$
$$p_2=\frac{[P_1\sin\alpha(t)+P_2\cos\alpha(t)]+m\dot{u}(t)x_2}{u(t)},\qquad d\tau=\frac{dt}{u^2(t)},$$

where $u(t)$ and $\alpha(t)$ are functions to be determined. From Eqs (8), we can see clearly that the transformation $(x_1,x_2,p_1,p_2)\longrightarrow(Q_1,Q_2,P_1,P_2)$ is nothing but a time-dependent canonical transformation with the generating function

$$F_2(x_1,x_2,P_1,P_2,t)=\frac{x_1P_1\cos\alpha(t)-x_1P_2\sin\alpha(t)+x_2P_1\sin\alpha(t)+x_2P_2\cos\alpha(t)}{u(t)}$$
$$+m\Big[\frac{\dot{u}(t)}{u(t)}\Big]\frac{x_1^2+x_2^2}{2}. \tag{9}$$

Using the discussion similar to that in Refs [7] and [8], we obtain the relation between the measures

$$\mathcal{D}x_1\mathcal{D}x_2Qp_1\mathcal{D}p_2=(u_iu_f)^{-1}\mathcal{D}Q_1\mathcal{D}Q_2\mathcal{D}P_1\mathcal{D}P_2, \tag{10}$$

by making use of Eqs (8) and (10), the propagator $K_2(x_{2f},x_{1f},t_f;x_{2i},x_{1i})$ can be rewritten as

$$K_2(x_{2f},x_{1f},t_f;x_{2i},x_{1i},t_i)=(u_iu_f)^{-1}\exp\Big\{\frac{im}{2\hbar}[u_f\dot{u}_f(Q_{2f}^2+Q_{1f}^2)-u_i\dot{u}_i(Q_{2i}^2+Q_{1i}^2)]\Big\}$$
$$\times\int\mathcal{D}Q_1\mathcal{D}Q_2\mathcal{D}P_1\mathcal{D}P_2\exp\Big\{\frac{i}{\hbar}\int_{\tau_i}^{\tau_f}\Big[P_1Q_1'+P_2Q_2'-\frac{(P_1^2+P_2^2)}{2m}\Big]d\tau\Big\}, \tag{11}$$

where

$$Q'_j = \frac{dQ_j}{d\tau} \quad (j=1,2), \qquad \alpha(t) = \int_0^t \Big[\frac{qB(t')}{2m}\Big] dt',$$

and $u(t)$ is the solution of the following equation

$$\ddot{u}(t) + b(t)u(t) = 0. \tag{12}$$

From Eq. (11), it is easy to see that the propagator

$$\int \mathcal{D}Q_1\mathcal{D}Q_2\mathcal{D}P_1\mathcal{D}P_2 \exp\Big\{\frac{i}{\hbar}\int_{\tau_i}^{\tau_f}\Big[P_1Q'_1 + P_2Q'_2 - \frac{P_1^2+P_2^2}{2m}\Big]d\tau\Big\}$$

is noting but that propagator for the two-dimensional free particle. Thus, we can get immediately

$$\begin{aligned}&\int \mathcal{D}Q_1\mathcal{D}Q_2\mathcal{D}P_1\mathcal{D}P_2 \exp\Big\{\frac{i}{\hbar}\int_{\tau_i}^{\tau_f}\Big[P_1Q'_1 + P_2Q'_2 - \frac{(P_1^2+P_2^2)}{2m}\Big]d\tau\Big\}\\ &= \frac{m}{2i\hbar\pi\tau_2}\exp\Big\{\frac{im}{2\hbar\tau_2}[(Q_{1f}-Q_{1i})^2 + (Q_{2f}-Q_{2i})^2]\Big\},\\ &\tau_2 = \int_{t_i}^{t_f} u^{-2}(t)dt,\end{aligned} \tag{13}$$

which leads to

$$\begin{aligned}K_2(x_{2f},x_{1f},t_f;x_{2i},x_{1i},t_i) &= \Big(\frac{m}{2i\hbar\pi u_i u_f \tau_2}\Big)\exp\Big\{\frac{im}{2\hbar}\Big[\frac{\dot{u}_f}{u_f}(x_{1f}^2+x_{2f}^2) - \frac{\dot{u}_i}{u_i}(x_{1i}^2+x_{2i}^2)\Big]\Big\}\\ &\times\exp\Big\{\frac{im}{2\hbar\tau_2}\Big[\frac{1}{u_f^2}(x_{1f}^2+x_{2f}^2) + \frac{1}{u_i^2}(x_{1i}^2+x_{2i}^2)\\ &- \frac{2}{(u_iu_f)}(x_{1f}x_{1i}+x_{2f}x_{2i})\cos\Big(\int_{t_i}^{t_f}\frac{qB(t)}{2m}dt\Big)\\ &+ \frac{2}{(u_iu_f)}(x_{1f}x_{2i}-x_{2f}x_{1i})\sin\Big(\int_{t_i}^{t_f}\frac{qB(t)}{2m}dt\Big)\Big]\Big\}.\end{aligned} \tag{14}$$

It is easy to check that the propagator

$$K(\vec{x}_f,t_f;\vec{x}_i,t_i) = K_1(x_{3f},t_f;x_{3i},t_i)K_2(x_{2f},x_{1f},t_f;x_{2i},x_{1i},t_i)$$

satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}K(\vec{x}_f,t_f;\vec{x}_i,t_i) = \hat{H}(t)K(\vec{x}_f,t_f;\vec{x}_i,t_i), \tag{15}$$

where $\hat{H}(t)$ is the quantum Hamiltonian corresponding to the classical Hamiltonian (1).

When $\omega(t) = \omega_c = eB_0/m, a(t) = b(t) = \omega_0^2$, the system studied in the present paper reduces to the system with Hamiltonian

$$H' = \sum_{j=1}^{3}\Big(\frac{p_j^2}{2m} + \frac{m\omega_0^2x_j^2}{2}\Big) + \Big(\frac{eB_0}{2m}\Big)(x_1p_2 - x_2p_1), \tag{16}$$

which was discussed in Ref. [10]. It is straightforward to show that $u(t) = v(t) = \cos\omega_0 t$ is a particular solution of Eqs (7) and (12). Therefore, the propagator corresponding to

Hamiltonian (16) is

$$K(\vec{x}_f,t;\vec{x}_i,0)=\left(\frac{m\omega_0}{2i\hbar\pi\sin\omega_0 t}\right)^{3/2}$$
$$\times\exp\left\{\left(\frac{im\omega_0}{2\hbar}\right)\left[\mathrm{ctg}\,\omega_0 t(x_{1f}^2+x_{2f}^2+x_{3f}^2+x_{1i}^2+x_{2i}^2+x_{3i}^2)\right.\right.$$
$$-\frac{2\cos(\omega_c t/2)}{\sin\omega_0 t}(x_{1i}x_{1f}+x_{2i}x_{2f}) \tag{17}$$
$$\left.\left.+\frac{2\sin(\omega_c t/2)}{\sin\omega_0 t}(x_{1f}x_{2i}-x_{1i}x_{2f})-2x_{3i}x_{3f}\right]\right\}.$$

This result is in agreement with that obtained in Ref. [10].

As a concluding remark, it is worth while to emphasize that using the propagator obtained in the present paper and the method used in Ref. [8], it is easy to construct the coherent states for the system with Hamiltonian (1).

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